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Remarks on the Sibony functions and pseudometrics

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Abstract. We discuss some basic properties of the Sibony functions and pseudometrics.

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1. Introduction. Let $G \subset \mathbb{C}^n$ be a domain. For $a \in G$ let

$$\mathcal{M}_G(a) := \{|f| : f \in \mathcal{O}(G,\mathbb{D}),\ f(a)=0\},$$

$$\mathcal{S}_G^{(p)}(a) := \{\sqrt[p]{u} :\ \ u : G \longrightarrow [0,1) : \log u \in \mathcal{PSH}(G),$$
$$u \in \mathcal{C}^p(\{a\}),\ \exists C>0 : u(z) \le C\|z-a\|^p,\ z \in G\}, \quad p \in \mathbb{N},$$

$$\mathcal{K}_G(a) := \{u :\ \ u : G \longrightarrow [0,1) : \log u \in \mathcal{PSH}(G),$$
$$\exists C>0 : u(z) \le C\|z-a\|,\ z \in G\},$$

where $\mathbb{D} \subset \mathbb{C}$ stands for the unit disc, $\mathcal{O}(G,\mathbb{D})$, resp. $\mathcal{PSH}(G)$ denote the set of all holomorphic functions on G having values in $\mathbb{D}$, resp. the set of all plurisubharmonic functions on G, and "$u \in \mathcal{C}^p(\{a\})$" means that u is of class $\mathcal{C}^p$ in a neighborhood of a (cf. [1, § 4.2]). Note that $\mathcal{S}_G^{(1)}(a)$ is different from $\mathcal{K}_G(a)$ (see Remark 2.1(c)). Put

$$\mathcal{S}_G(a) := \mathcal{S}_G^{(2)}(a) = \{\sqrt{u} :\ \ u : G \longrightarrow [0,1) : \log u \in \mathcal{PSH}(G),$$
$$u \in \mathcal{C}^2(\{a\}),\ u(0)=0\}.$$

Obviously, $\mathcal{M}_G(a) \subset \mathcal{S}_G(a) \subset \mathcal{K}_G(a)$ and $\mathcal{S}_G^{(p)}(a) \subset \mathcal{K}_G(a)$, $p \in \mathbb{N}$. If $\mathcal{F} \in \{\mathcal{M}, \mathcal{S}^{(p)}, \mathcal{K}\}$, then we define:

$$d_G^{\mathcal{F}}(a,z) := \sup\{v(z) :\ v \in \mathcal{F}_G(a)\}, \quad a,z \in G,$$

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$$\delta_G^{\mathcal{F}}(a;X) := \sup\Big\{\limsup_{\lambda\to 0}\frac{v(a+\lambda X)}{|\lambda|} : v\in\mathcal{F}_G(a)\Big\},\quad a\in G,\ X\in\mathbb{C}^n.$$

For $\mathcal{F}\in\{\mathcal{M},\mathcal{S},\mathcal{K}\}$ the families $(d_G^{\mathcal{F}})_G$ and $(\delta_G^{\mathcal{F}})_G$ are *holomorphically contractible*, i.e.

- $d_{\mathbb{D}}^{\mathcal{F}}(0,t)=t$, $t\in[0,1)$, $\delta_{\mathbb{D}}^{\mathcal{F}}(0;1)=1$;
- for any domains $G\subset\mathbb{C}^n$, $D\subset\mathbb{C}^m$ and for any holomorphic mapping $F:G\longrightarrow D$ we have

$$d_D^{\mathcal{F}}(F(a),F(z))\le d_G^{\mathcal{F}}(a,z),\quad a,z\in G, \tag{1.1}$$

$$\delta_D^{\mathcal{F}}(F(a);F'(a)(X))\le \delta_G^{\mathcal{F}}(a;X),\quad a\in G,\ X\in\mathbb{C}^n. \tag{1.2}$$

In particular, the families $(d_G^{\mathcal{F}})_G$ and $(\delta_G^{\mathcal{F}})_G$ are invariant under biholomorphic mappings.

If $\mathcal{F}=\mathcal{M}$, then we get the *Möbius pseudodistance* $\boldsymbol{m}_G := d_G^{\mathcal{M}}$ and the *Carathéodory–Reiffen pseudometric* $\boldsymbol{\gamma}_G := \delta_G^{\mathcal{M}}$. It is known that

$$\boldsymbol{\gamma}_G(a,z)=\lim_{\lambda\to 0}\frac{\boldsymbol{m}_G(a,a+\lambda X)}{|\lambda|}=\max\{|f'(z)(X)| : f\in\mathcal{O}(G,\mathbb{D}),\ f(a)=0\}. \tag{1.3}$$

If $\mathcal{F}=\mathcal{S}$, then we get the *Sibony function* $\boldsymbol{s}_G := d_G^{\mathcal{S}}$ and the *Sibony pseudometric* $\boldsymbol{S}_G := \delta_G^{\mathcal{S}}$. It is known that

$$\boldsymbol{S}_G(a;X)=\sup\{\sqrt{\mathcal{L}u(a;X)} : u\in\mathcal{S}_G(a)\},$$

where $\mathcal{L}u(a;X) := \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j\partial\bar z_k}(a)X_j\overline{X}_k$ is the *Levi form* (cf. [1, Proposition 4.2.16]). In particular, $\boldsymbol{S}_G(a;\cdot)$ is a $\mathbb{C}$-seminorm.

If $\mathcal{F}=\mathcal{K}$, then we get the *pluricomplex Green function* $\boldsymbol{g}_G := d_G^{\mathcal{K}}$ and the *Azukawa pseudometric* $\boldsymbol{A}_G := \delta_G^{\mathcal{K}}$. It is known that $\boldsymbol{g}_G(a,\cdot)\in\mathcal{K}_G(a)$, $\log\boldsymbol{A}_G(a;\cdot)\in\mathcal{PSH}(\mathbb{C}^n)$, and

$$\boldsymbol{A}_G(a;X)=\limsup_{\lambda\to 0}\frac{\boldsymbol{g}_G(a,a+\lambda X)}{|\lambda|}\quad\text{(cf. [JP 2013, Lemma 4.2.3]).} \tag{1.4}$$

If $\mathcal{F}=\mathcal{S}^{(p)}$, $p\neq 2$, then we get the *higher order Sibony function* $\boldsymbol{s}_G^{(p)} := d_G^{\mathcal{S}^{(p)}}$ and the *higher order Sibony pseudometric* $\boldsymbol{S}_G^{(p)} := \delta_G^{\mathcal{S}^{(p)}}$.

Recall that many properties of complex domains/manifolds are encoded in $\boldsymbol{m}_G$, $\boldsymbol{\gamma}_G$, $\boldsymbol{g}_G$, and $\boldsymbol{A}_G$. Therefore, these functions have been important tools to investigate various geometric problems in several complex analysis. Their study over the last years has led to the point that their basic properties are now well understood. To extent the family of invariant functions higher order functions were introduced and studied, e.g. the higher Carathéodory-Reiffen pseudometrics. But in contrast to that little is known on properties of $\boldsymbol{S}_G$ and almost nothing on $\boldsymbol{s}_G^{(p)}$, $p\in\mathbb{N}$, and $\boldsymbol{S}_G^{(p)}$, $p\neq 2$.

The main aim of this note is to show that the basic properties of $\boldsymbol{s}_G^{(p)}$ and $\boldsymbol{S}_G^{(p)}$ differ essentially from the corresponding properties of $\boldsymbol{m}_G$, $\boldsymbol{g}_G$, $\boldsymbol{\gamma}_G$, and $\boldsymbol{A}_G$. Surprisingly, as we will see, many properties of the so far studied invariant functions fail to hold.

2. Holomorphic contractibility.

Remark 2.1. (a) $\mathcal{S}_G^{(p)}(a) = \{\sqrt[p]{u} : \ u : G \longrightarrow [0,1) : \log u \in \mathcal{PSH}(G),\ u \in \mathcal{C}^p(\{a\}),\ \operatorname{ord}_a u \geq p\}$, where $\operatorname{ord}_a u$ denotes the order of zero of u at a.

(b) In view of the Taylor formula, we have

$$\boldsymbol{S}_G^{(p)}(a;X) = \sup\left\{\left(\tfrac{1}{p!}|u^{(p)}(a)(X)|\right)^{1/p} : \sqrt[p]{u} \in \mathcal{S}_G^{(p)}(a)\right\}, \quad a \in G,\ X \in \mathbb{C}^n,$$

where $u^{(p)}(a) : \mathbb{C}^n \longrightarrow \mathbb{R}$ stands for the p-th Fréchet differential of u at a.

(c) In view of (b), we get $\boldsymbol{S}_G^{(p)}(a;\cdot) \equiv 0$ for p odd. In particular, $\boldsymbol{S}_{\mathbb{D}}^{(1)}(0;1) = 0 < 1 = \boldsymbol{A}_{\mathbb{D}}(0;1)$.

(d) $\boldsymbol{s}_G^{(p)} \leq \boldsymbol{g}_G$, $\boldsymbol{S}_G^{(p)} \leq \boldsymbol{A}_G$. In particular, $\boldsymbol{s}_{\mathbb{D}}^{(p)}(0,\lambda) \leq \boldsymbol{g}_{\mathbb{D}}(0,\lambda) = |\lambda|$, $\boldsymbol{S}_{\mathbb{D}}^{(p)}(0;1) \leq \boldsymbol{A}_{\mathbb{D}}(0;1) = 1$.

(e) If $\boldsymbol{g}_G^{p+\varepsilon}(a,\cdot) \in \mathcal{C}^p(\{a\})$ for $0 < \varepsilon \ll 1$, then $\boldsymbol{g}_G^{1+\varepsilon/p}(a,\cdot) \in \mathcal{S}_G^{(p)}(a)$. Consequently, $\boldsymbol{s}_G^{(p)}(a,\cdot) = \boldsymbol{g}_G(a,\cdot)$. In particular, $\boldsymbol{s}_{\mathbb{D}}^{(p)}(0,\lambda) = |\lambda|$, $\lambda \in \mathbb{D}$.

(f) If $\boldsymbol{g}_G^{2p}(a,\cdot) \in \mathcal{C}^{2p}(\{a\})$, then $\boldsymbol{S}_G^{(2p)}(a;\cdot) = \boldsymbol{A}_G(a;\cdot)$. In particular, $\boldsymbol{S}_{\mathbb{D}}^{(2p)}(0;1) = 1$.

(g) If $F : G \longrightarrow D$ is holomorphic, then $v \circ F \in \mathcal{S}_G^{(p)}(a)$ for every $v \in \mathcal{S}_D^{(p)}(F(a))$. Consequently, the family $(\boldsymbol{s}_G^{(p)})_G$ (resp. $(\boldsymbol{S}_G^{(p)})_G$) satisfies (1.1) (resp. (1.2)).

(h) The families $(\boldsymbol{s}_G^{(p)})_G$ and $(\boldsymbol{S}_G^{(2p)})_G$ are holomorphically contractible. They will be the main objects of our investigation in the sequel.

(i) $\boldsymbol{m}_G \leq \boldsymbol{s}_G^{(p)} \leq \boldsymbol{g}_G$ and $\boldsymbol{\gamma}_G \leq \boldsymbol{S}_G^{(2p)} \leq \boldsymbol{A}_G$.

3. Upper semicontinuity. It is known that for $\mathcal{F} \in \{\mathcal{M},\mathcal{K}\}$ the functions $G \times G \ni (z,w) \longmapsto d_G^{\mathcal{F}}(z,w)$ and $G \times \mathbb{C}^n \ni (z,X) \longmapsto \delta_G^{\mathcal{F}}(z;X)$ are upper semicontinuous (cf. [1, Propositions 2.6.1, 2.7.1(c), 4.2.10(g,k)]). We will prove that in general the functions $\boldsymbol{s}_G^{(p)}(\cdot,z^0)$ and $\boldsymbol{S}_G^{(2p)}(\cdot;X^0)$ are not upper semicontinuous (Examples 3.1, 3.3).

Recall that $\boldsymbol{S}_G(a;\cdot)$ is a seminorm and therefore it is continuous. We do not know whether the functions $\boldsymbol{s}_G(a,\cdot)$, $p \in \mathbb{N}$, and $\boldsymbol{S}_G^{(2p)}(a;\cdot)$, $p \geq 2$, are upper semicontinuous.

Example 3.1. (cf. [1, Example 4.2.18]) Let

$$G := \{(z_1,z_2,z_3) \in \mathbb{C}^3 : |z_1|e^{\varphi(z_2,z_3)} < 1\}$$

with

$$\varphi(\xi,\eta) := \sum_{k=1}^{\infty} \lambda_k \log\Big(\frac{|\xi - a_k|^2 + |\eta|}{k}\Big), \quad (\xi,\eta) \in \mathbb{C}^2,$$

where $(a_k)_{k=1}^{\infty} \subset \mathbb{D}\backslash\{0\}$ is a dense subset of $\mathbb{D}$ and $(\lambda_k)_{k=1}^{\infty} \subset (0,1]$ are chosen so that $\varphi(0,0) > -\infty$ and $\varphi \in \mathcal{C}^\infty(\mathbb{C} \times \mathbb{C}_*)$, where $\mathbb{C}_* := \mathbb{C}\backslash\{0\}$. Note that G is a pseudoconvex Hartogs domain.

Let $c_t := (0,0,t) \in G$, $t > 0$, $z^0 := (b,0,0) \in G$ with $b \neq 0$, and let $X^0 := (1,0,0)$. We will show that

$$\boldsymbol{s}_G^{(p)}(0,z^0) = 0 < |b|e^{\varphi(0,0)} \leq \boldsymbol{s}_G^{(p)}(c_t,z^0),$$

$$\boldsymbol{S}_G^{(2p)}(0;X^0)=0<e^{\varphi(0,0)}\le\boldsymbol{S}_G^{(2p)}(c_t;X^0),\quad 0<t\ll 1,$$

which shows that the functions $\boldsymbol{s}_G^{(p)}(\cdot,z^0)$ and $\boldsymbol{S}_G^{(2p)}(\cdot;X^0)$ are not upper semicontinuous at 0.

Indeed, the function $G\ni(z_1,z_2,z_3)\overset{v}{\longmapsto}(|z_1|e^{\varphi(z_2,z_3)})^{1+\varepsilon/p}$ belongs to $\mathcal{S}_G^{(p)}(c_t)$ for all $\varepsilon>0$ and $t>0$. Hence, $\boldsymbol{s}_G^{(p)}(c_t,z^0)\ge|b|e^{\varphi(0,0)}>0$. Analogously, the function $G\ni(z_1,z_2,z_3)\overset{v}{\longmapsto}|z_1|e^{\varphi(z_2,z_3)}$ belongs to $\mathcal{S}_G^{(2p)}(c_t)$ for all $t>0$. Hence, $\boldsymbol{S}_G^{(2p)}(c_t;X^0)\ge\limsup_{\lambda\to0}\frac{v(c_t+\lambda X^0)}{|\lambda|}=e^{\varphi(0,t)}\ge e^{\varphi(0,0)}>0$.

On the other hand, let $\sqrt[p]{u}\in\mathcal{S}_G^{(p)}(0)$ (resp. $\sqrt[2p]{u}\in\mathcal{S}_G^{(2p)}(0)$). Since $\mathbb{C}\times\{a_k\}\times\{0\}\subset G$, we get $u(z_1,a_k,0)=\mathrm{const}(k)$, $z_1\in\mathbb{C}$, $k\in\mathbb{N}$. Since $\{0\}\times\mathbb{C}\times\{0\}\subset G$, we get $u(0,z_2,0)=\mathrm{const}=u(0)=0$, $z_2\in\mathbb{C}$. Thus, $u(z_1,a_k,0)=0$, $z_1\in\mathbb{C}$, $k\in\mathbb{N}$. Since $u\in\mathcal{C}^p(\{0\})$ (resp. $u\in\mathcal{C}^{2p}(\{0\})$), we conclude that $u=0$ in $U\times\{0\}$, where U is a neighborhood of $(0,0)$. Since $\log u\in\mathcal{PSH}(G)$, we get $u(z_1,z_2,0)=0$ for all $(z_1,z_2,0)\in G$. Consequently, $\boldsymbol{s}_G^{(p)}(0,z^0)=0$ (resp. $\boldsymbol{S}_G^{(2p)}(0;X^0)=0$).

Example 3.2. In view of Example 3.1, one could expect that perhaps the families $(\boldsymbol{s}_G^{(p)*})_G$ and/or $(\boldsymbol{S}_G^{(2p)*})_G$ are holomorphically contractible, where $\boldsymbol{s}_G^{(p)*}:=(\boldsymbol{s}_G^{(p)})^*$, $\boldsymbol{S}_G^{(2p)*}:=(\boldsymbol{S}_G^{(2p)})^*$, and * denotes the upper semicontinuous regularization. We will prove that unfortunately they are not holomorphically contractible.

Keep the notation from Example 3.1. Let

$$D:=\{(z_1,z_2)\in\mathbb{C}^2:(z_1,z_2,0)\in G\},\quad D\ni(z_1,z_2)\overset{F}{\longmapsto}(z_1,z_2,0)\in G.$$

Then $\boldsymbol{s}_G^{(p)*}(0,z^0)\ge\limsup_{t\to0+}\boldsymbol{s}_G^{(p)}(c_t,z^0)\ge|b|e^{\varphi(0,0)}>0$ and $\boldsymbol{S}_G^{(2p)*}(0;X^0)\ge\limsup_{t\to0+}\boldsymbol{S}_G^{(2p)}(c_t;X^0)\ge e^{\varphi(0,0)}>0$.

On the other hand, let $w^0\in D\cap(\mathbb{C}\times\mathbb{D})$ and let $\sqrt[p]{u}\in\mathcal{S}_D^{(p)}(\{w^0\})$ (resp. $\sqrt[2p]{u}\in\mathcal{S}_D^{(2p)}(\{w^0\})$). Since $\mathbb{C}\times\{a_k\}\subset D$, we get $u(z_1,a_k)=\mathrm{const}(k)$, $z_1\in\mathbb{C}$, $k\in\mathbb{N}$. Since $\{0\}\times\mathbb{C}\subset D$, we get $u(0,z_2)=\mathrm{const}=u(0,0)$, $z_2\in\mathbb{C}$. Thus, $u(z_1,a_k)=\mathrm{const}$, $z_1\in\mathbb{C}$, $k\in\mathbb{N}$. Since $u\in\mathcal{C}^p(\{w^0\})$ (resp. $u\in\mathcal{C}^{2p}(\{w^0\})$), we conclude that $u=0$ in $U\times\{0\}$, where U is a neighborhood of w^0. Hence, since $\log u\in\mathcal{PSH}(G)$, we get $u(z_1,z_2)=0$ for all $(z_1,z_2)\in D$. Consequently, $\boldsymbol{s}_D^{(p)}=0$ on $(D\cap(\mathbb{C}\times\mathbb{D}))\times D$ (resp. $\boldsymbol{S}_D^{(2p)}=0$ on $(D\cap(\mathbb{C}\times\mathbb{D}))\times\mathbb{C}^2$). In particular, $\boldsymbol{s}_D^{(p)*}(0,(b,0))=0$ (resp. $\boldsymbol{S}_D^{(2p)*}(0;(1,0))=0$) and therefore

$$\boldsymbol{s}_G^{(p)*}(F(0,0),F(b,0))>0=\boldsymbol{s}_D^{(p)*}((0,0),(b,0)),$$
$$\boldsymbol{S}_G^{(2p)*}(F(0,0);F'(0,0)(1,0))>0=\boldsymbol{S}_D^{(2p)*}((0,0);(1,0)).$$

Example 3.3. For $n\ge2$ and $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n\setminus\{0\}$ let

$$\boldsymbol{D}_\alpha:=\{z\in\mathbb{C}^n(\alpha):|z^\alpha|:=|z_1|^{\alpha_1}\cdots|z_n|^{\alpha_n}<1\},$$

where $\mathbb{C}^n(\alpha):=\{(z_1,\dots,z_n)\in\mathbb{C}^n:\forall j\in\{1,\dots,n\}:(\alpha_j<0\Longrightarrow z_j\ne0)\}$. Note that $\boldsymbol{D}_\alpha$ is a pseudoconvex Reinhardt domain. For $a=(a_1,\dots,a_n)\in\boldsymbol{D}_\alpha$ define

$$\Xi(a) := \{j \in \{1, \dots, n\} : \alpha_j > 0,\ a_j = 0\},$$
$$r(a) := \begin{cases} 1 & \text{if } \sigma(a) = 0 \\ \sum_{j \in \Xi(a)} \alpha_j & \text{if } \sigma(a) \geq 1 \end{cases}, \quad \sigma(a) := \#\Xi(a), \quad \mu(a) := \min\{\alpha_j : j \in \Xi(a)\}.$$

Note that if $\sigma(a) = 1$, then $r(a) = \mu(a)$.

The following results are known (cf. [1, §§ 6.2, 6.3], and [2, Theorem 1]).

- If $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$ are relatively prime, then

$$\boldsymbol{m}_{\boldsymbol{D}_\alpha}(a, z) = \boldsymbol{m}_{\mathbb{D}}(a^\alpha, z^\alpha), \quad \boldsymbol{g}_{\boldsymbol{D}_\alpha}(a, z) = \big(\boldsymbol{m}_{\mathbb{D}}(a^\alpha, z^\alpha)\big)^{1/r},$$
$$\boldsymbol{A}_{\boldsymbol{D}_\alpha}(a; X) = \Big(\boldsymbol{\gamma}_{\mathbb{D}}(a^\alpha; \tfrac{1}{r!} \prod_{j \notin \Xi(a)} a_j^{\alpha_j} \cdot \prod_{j \in \Xi(a)} X_j^{\alpha_j})\Big)^{1/r}, \quad r = r(a),$$
$$\boldsymbol{s}_{\boldsymbol{D}_\alpha}(a, z) = \begin{cases} \boldsymbol{m}_{\mathbb{D}}(a^\alpha, z^\alpha) & \text{if } \sigma(a) = 0 \\ |z^\alpha|^{1/\mu(a)} & \text{if } \sigma(a) \geq 1 \end{cases},$$
$$\boldsymbol{S}_{\boldsymbol{D}_\alpha}(a; X) = \begin{cases} \boldsymbol{A}_{\boldsymbol{D}_\alpha}(a; X) & \text{if } \sigma(a) \leq 1 \\ 0 & \text{if } \sigma(a) \geq 2 \end{cases}, \quad a, z \in \boldsymbol{D}_\alpha,\ X \in \mathbb{C}^n.$$

- If $\alpha \notin \mathbb{R} \cdot \mathbb{Z}^n$, then

$$\boldsymbol{m}_{\boldsymbol{D}_\alpha} \equiv 0, \quad \boldsymbol{g}_{\boldsymbol{D}_\alpha}(a, z) = \begin{cases} 0 & \text{if } \sigma(a) = 0 \\ |z^\alpha|^{1/r} & \text{if } \sigma(a) \geq 1 \end{cases},$$
$$\boldsymbol{A}_{\boldsymbol{D}_\alpha}(a; X) = \begin{cases} 0 & \text{if } \sigma(a) = 0 \\ \big(\prod_{j \notin \Xi(a)} |a_j|^{\alpha_j} \cdot \prod_{j \in \Xi(a)} |X_j|^{\alpha_j}\big)^{1/r} & \text{if } \sigma(a) \geq 1 \end{cases}, \quad r = r(a),$$
$$\boldsymbol{s}_{\boldsymbol{D}_\alpha}(a, z) = \begin{cases} 0 & \text{if } \sigma(a) = 0 \\ |z^\alpha|^{1/\mu(a)} & \text{if } \sigma(a) \geq 1 \end{cases},$$
$$\boldsymbol{S}_{\boldsymbol{D}_\alpha}(a; X) = \begin{cases} \boldsymbol{A}_{\boldsymbol{D}_\alpha}(a; X) & \text{if } \sigma(a) = 1 \\ 0 & \text{if } \sigma(a) \neq 1 \end{cases}, \quad a, z \in \boldsymbol{D}_\alpha,\ X \in \mathbb{C}^n.$$

In particular, if $n = 3$ and $\alpha = (1, 2, 2)$, then $\boldsymbol{s}_{\boldsymbol{D}_\alpha}((0, 0, 0), z) = |z^\alpha|$ and $\boldsymbol{s}_{\boldsymbol{D}_\alpha}((1/k, 0, 0), z) = |z^\alpha|^{1/2}$, $k \in \mathbb{N}$. Thus, the function $\boldsymbol{s}_{\boldsymbol{D}_\alpha}(\cdot, z^0)$ is not upper semicontinuous at $(0, 0, 0)$ for all $z^0 = (z_1^0, z_2^0, z_3^0) \in \boldsymbol{D}_\alpha$ with $z_1^0 z_2^0 z_3^0 \neq 0$.

Notice that using the above effective formulas one may easily construct many other counterexamples.

Example 3.4. Keep the notation from Example 3.3. Assume that $\alpha_1, \dots, \alpha_n \in \mathbb{R}_* := \mathbb{R} \backslash \{0\}$, $a_1 \cdots a_s \neq 0$, $a_{s+1} = \cdots = a_n = 0$, $s := n - \sigma(a)$. In particular, $\alpha_{s+1}, \dots, \alpha_n > 0$.

First observe that if $\sigma(a) \leq 1$, then $\boldsymbol{g}_{\boldsymbol{D}_\alpha}^{p+\varepsilon}(a, \cdot) \in \mathcal{C}^p(\{a\})$ and consequently $\boldsymbol{s}_{\boldsymbol{D}_\alpha}^{(p)}(a, \cdot) = \boldsymbol{g}_{\boldsymbol{D}_\alpha}(a, \cdot)$ (Remark 2.1(e)). Similarly, if $\sigma(a) \leq 1$, then $\boldsymbol{g}_{\boldsymbol{D}_\alpha}^{2p}(a, \cdot) \in \mathcal{C}^\infty(\{a\})$ and consequently $\boldsymbol{S}_{\boldsymbol{D}_\alpha}^{(2p)}(a; \cdot) = \boldsymbol{A}_{\boldsymbol{D}_\alpha}(a; \cdot)$ (Remark 2.1(f)). Problems start when $\sigma(a) \geq 2$. We do not know effective formulas for $\boldsymbol{s}_{\boldsymbol{D}_\alpha}^{(p)}(a, \cdot)$, $p \neq 2$, and $\boldsymbol{S}_{\boldsymbol{D}_\alpha}^{(2p)}(a; \cdot)$, $p \geq 2$. To illustrate problems we discuss some particular cases.

- Assume that $\sigma(a) \geq 1$ and $k_j := \frac{p\alpha_j}{r(a)} \in \mathbb{N}$, $j = s+1, \dots, n$. Then

$$\boldsymbol{g}_{\boldsymbol{D}_\alpha}^{2p}(a, z) = \prod_{j=1}^{s} |z_j|^{2p\alpha_j/r(a)} \cdot \prod_{j=s+1}^{n} |z_j|^{2k_j}$$

and consequently $\boldsymbol{g}_{\boldsymbol{D}_\alpha}^{2p}(a, \cdot) \in \mathcal{C}^\infty(\{a\})$ which gives $\boldsymbol{S}_{\boldsymbol{D}_\alpha}^{(2p)}(a; \cdot) = \boldsymbol{A}_{\boldsymbol{D}_\alpha}(a; \cdot)$. For example, if $n = 2$, $\alpha = (1, 1)$, $a = (0, 0)$, then $\boldsymbol{S}_{\boldsymbol{D}_\alpha}^{(4k)}(a; X) = |X_1 X_2|^{1/2}$, $k \in \mathbb{N}$.

- Assume that $\sigma(a) \geq 1$ and there exists a $j_0 \in \{s+1, \dots, n\}$ such that $\frac{2p\alpha_{j_0}}{r(a)} \notin \mathbb{N}$. Then $\boldsymbol{S}_{\boldsymbol{D}_\alpha}^{(2p)}(a; \cdot) \equiv 0$.

Indeed, we may assume that $j_0 = n$. Let $r := r(a)$ and let $k \in \mathbb{N}_0$ be such that $k < \frac{2p\alpha_n}{r} < k+1$. In view of Remark 2.1(b), we have to prove that $u^{(2p)}(a) \equiv 0$ for all $\sqrt[2p]{u} \in \mathcal{S}_{\boldsymbol{D}_\alpha}^{(2p)}(a)$. Fix such a u and suppose that $u^{(2p)}(a)(X^0) \neq 0$ for some $X^0 \neq 0$. We have

$$\begin{aligned}\Big(\frac{1}{(2p)!}|u^{(2p)}(a)(X)|\Big)^{1/2p} &\leq \boldsymbol{S}_{\boldsymbol{D}_\alpha}^{(2p)}(a; X)\\ &\leq \boldsymbol{A}_{\boldsymbol{D}_\alpha}(a; X) = \Big(\prod_{j=1}^{s} |a_j|^{\alpha_j} \cdot \prod_{j=s+1}^{n} |X_j|^{\alpha_j}\Big)^{1/r}.\end{aligned}$$

Write $u^{(2p)}(a)(X_1^0, \dots, X_{n-1}^0, tX_n^0) = A_d t^d + \dots + A_0$, $t \in \mathbb{R}$, with $A_d \neq 0$. We have $|A_d t^d + \dots + A_0| \leq \text{const}\,|t|^{2p\alpha_n/r}$, $t \in \mathbb{R}$. Taking $t \longrightarrow \infty$ we get $d \leq k$. On the other hand, taking $t \longrightarrow 0$ we get $A_d = 0$; a contradiction.

For example let $n = 2$, $\alpha = (q, 1)$, $a = (0, 0)$, where

$$0 < q \notin \{\tfrac{2p-k}{k} : k = 1, \dots, 2p-1\} \cap \{\tfrac{k}{2p-k} : k = 1, \dots, 2p-1\}.$$

Then $\boldsymbol{S}_{\boldsymbol{D}_\alpha}^{(2p)}(a; \cdot) \equiv 0$.

- As a consequence, we conclude that for every $s \in \{0, \dots, n-2\}$ there exists a set C_s dense in $\mathbb{R}_*^s \times \mathbb{R}_{>0}^{n-s}$ ($\mathbb{R}_{>0} := (0, +\infty)$) such that for any $\alpha \in C_s$, $a \in \boldsymbol{D}_\alpha \cap (\mathbb{C}_*^s \times \{0\}^{n-s})$, and $p \in \mathbb{N}$ we have $\boldsymbol{S}_{\boldsymbol{D}_\alpha}^{(2p)}(a; \cdot) \equiv 0$.

Indeed, we may put

$$C_s := (\mathbb{R}_*^s \times \mathbb{R}_{>0}^{n-s}) \setminus \bigcup_{\substack{p,k \in \mathbb{N}:\ k < 2p\\ j \in \{s+1, \dots, n\}}} \{\alpha \in \mathbb{R}^n : 2p\alpha_j = k(\alpha_{s+1} + \dots + \alpha_n)\}.$$

Now we turn to discuss a special case where $G \subset \mathbb{C}^n$ is a complete n-circled domain (Example 3.5).

Example 3.5. Let $G \subset \mathbb{C}^n$ be a complete n-circled domain, i.e. for any $z = (z_1, \dots, z_n) \in G$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \overline{\mathbb{D}}^n$, the point $\lambda \cdot z := (\lambda_1 z_1, \dots, \lambda_n z_n)$ belongs to G.

(a) Since $\boldsymbol{s}_G^{(p)}(0, \cdot) \leq \boldsymbol{g}_G(0, \cdot)$ and the Green function is upper semicontinuous, the function $\boldsymbol{s}_G^{(p)}(0, \cdot)$ is continuous at 0.

(b) The function $\boldsymbol{s}_G^{(p)}(0, \cdot)$ is upper semicontinuous in the domain $G \setminus \boldsymbol{V}_0$, where $\boldsymbol{V}_0 := \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 \cdots z_n = 0\}$.

Indeed, let $M := \{a \in G : \boldsymbol{s}_G^{(p)}(0, \cdot)$ is not upper semicontinuous at $a\}$.

Since $\boldsymbol{s}_G^{(p)}(0,\cdot)$ is invariant under n-rotations (i.e. under mappings $G \ni z \longmapsto \lambda \cdot z \in G$, $\lambda \in \mathbb{T}^n$, where $\mathbb{T} := \partial\mathbb{D}$), the set M is also invariant under n-rotations. It is known that M is pluripolar, i.e. there exists a $v \in \mathcal{PSH}(\mathbb{C}^n)$, $v \not\equiv -\infty$, such that $M \subset v^{-1}(-\infty)$ (cf. [3, Theorem 4.7.6]). Suppose that $a = (a_1, \dots, a_n) \in M \setminus \boldsymbol{V}_0$. Then $v(\lambda \cdot a) = -\infty$ for all $\lambda \in \mathbb{T}^n$. Consequently, by the maximum principle for plurisubharmonic functions, $v(z_1, \dots, z_n) = -\infty$ for all $|z_j| \le |a_j|$, $j = 1, \dots, n$. Hence, $v \equiv -\infty$; a contradiction.

(c) Let $a = (0, \dots, 0, a_{s+1}, \dots, a_n) =: (0, b) \in G \cap \boldsymbol{V}_0$, $1 \le s \le n-1$, $a_{s+1} \cdots a_n \ne 0$. Define $D := \{\zeta \in \mathbb{C}^{n-s} : (\underbrace{0, \dots, 0}_{s\times}, \zeta) \in G\}$. Note that D is a complete $(n-s)$-circled domain with $b \in D$. Let $\mathfrak{h}_D$ denote the Minkowski functional of D ($\mathfrak{h}_D(\zeta) := \inf\{1/t : t > 0,\ t\zeta \in D\}, \zeta \in \mathbb{C}^{n-s}$). Observe that $\mathfrak{h}_D$ is continuous (because D is $(n-s)$-circled).

Assume that $\boldsymbol{s}_D^{(p)}(0, b) = \mathfrak{h}_D(b)$. Then the function $\boldsymbol{s}_G^{(p)}(0,\cdot)$ is upper semicontinuous at a.

Indeed, let $0 < R < 1$ and $k > 0$ be such that $\mathfrak{h}_D(b) < R$ and $\|b\| < k$. Note that $\{\zeta \in D : \mathfrak{h}_D(\zeta) < R,\ \|\zeta\| < k\} \subset\subset D$. Consequently, there exists an $\varepsilon > 0$ such that $U := \{(z', z'') \in \mathbb{C}^n : \|z'\| < \varepsilon,\ \mathfrak{h}_D(z'') < R,\ \|z''\| < k\} \subset G$. Then for $z = (z', z'') \in U$ we have

$$\boldsymbol{s}_G^{(p)}(0, z) \le \boldsymbol{s}_U^{(p)}(0, z) \le \boldsymbol{g}_U(0, z) \le \max\Big\{\frac{\|z'\|}{\varepsilon}, \frac{\mathfrak{h}_D(z'')}{R}, \frac{\|z''\|}{k}\Big\}.$$

Hence,

$$\limsup_{z\to a} \boldsymbol{s}_G^{(p)}(0, z) \le \limsup_{(z',z'')\to(0,b)} \max\Big\{\frac{\|z'\|}{\varepsilon}, \frac{\mathfrak{h}_D(z'')}{R}, \frac{\|z''\|}{k}\Big\} = \max\Big\{\frac{\mathfrak{h}_D(b)}{R}, \frac{\|b\|}{k}\Big\}.$$

Letting $R \longrightarrow 1$ and $k \longrightarrow +\infty$ we get $\limsup_{z\to a} \boldsymbol{s}_G^{(p)}(0, z) \le \mathfrak{h}_D(b)$.

On the other side, since the projection $\mathbb{C}^s \times \mathbb{C}^{n-s} \ni (z', z'') \longmapsto z'' \in D$ is well-defined, we get $\mathfrak{h}_D(b) = \boldsymbol{s}_D^{(p)}(0, b) \le \boldsymbol{s}_G^{(p)}(0, a)$.

(d) Observe that $\boldsymbol{s}_D^{(p)}(0, b) = \mathfrak{h}_D(b)$ in the case where D is convex. If $s = n-1$, then D is either a disc or the whole $\mathbb{C}$. Thus, if $s = n-1$, then the function $\boldsymbol{s}_G^{(p)}(0,\cdot)$ is upper semicontinuous at each point $a \in \boldsymbol{V}_0$ of the form $a = (0, \dots, 0, a_j, 0, \dots, 0) \in G$.

(e) Consequently, if $n = 2$, then the function $\boldsymbol{s}_G^{(p)}(0,\cdot)$ is globally upper semicontinuous.

(f) If G is bounded, then the function $\boldsymbol{s}_G^{(p)}(0,\cdot)$ is globally upper semicontinuous.

Indeed, we proceed by induction on $n \ge 2$. The case $n = 2$ is solved in (e). Suppose the result is true for $n - 1 \ge 2$. Let $a = (a_1, \dots, a_n) \in G \cap \boldsymbol{V}_0$ (see Example 3.5 (b)). We may assume that $a_{n-1} \ne 0$, $a_n = 0$. Define $D := \{z' \in \mathbb{C}^{n-1} : (z', 0) \in G\}$; D is a bounded complete $(n-1)$-circled domain. Thus, by the inductive assumption, $\boldsymbol{s}_D^{(p)}(0,\cdot)$ is upper semicontinuous. Since G is bounded, for every $0 < r < 1$ with $a \in rG$ there exists an $\varepsilon > 0$ such that $(rD) \times \mathbb{D}(\varepsilon) \subset\subset G$. Suppose that $G \subset \mathbb{D}^n(R)$ and let $\eta > 0$

be such that $rR|\frac{z_n}{z_{n-1}}| < \varepsilon$ for $z \in U := \{z = (z', z_n) \in a + \mathbb{D}^n(\eta) : z' \in rD\}$. For $z \in U$ consider the holomorphic mapping $F_z : rD \longrightarrow G$, $F_z(w) := (w, w_{n-1}\frac{z_n}{z_{n-1}})$. We have $\boldsymbol{s}_G^{(p)}(0, F_z(w)) \le \boldsymbol{s}_{rD}^{(p)}(0, w) = \boldsymbol{s}_D^{(p)}(0, w/r)$. In particular, $\boldsymbol{s}_G^{(p)}(0, z) = \boldsymbol{s}_G^{(p)}(0, F_z(z')) \le \boldsymbol{s}_D^{(p)}(0, z'/r)$. Thus, $\limsup_{z\to a} \boldsymbol{s}_G^{(p)}(0, z) \le \limsup_{z\to a} \boldsymbol{s}_D^{(p)}(0, z'/r) = \boldsymbol{s}_D^{(p)}(0, a'/r)$. Letting $r \longrightarrow 1-$ (and using once again the upper semicontinuity of $\boldsymbol{s}_D^{(p)}(0, \cdot)$ we get $\limsup_{z\to a} \boldsymbol{s}_G^{(p)}(0, z) \le \boldsymbol{s}_D^{(p)}(0, a') \le \boldsymbol{s}_G^{(p)}(0, a)$ (cf. Example 3.5 (c)).

Note that if $n \ge 3$ and D is unbounded, then it is not known whether the function $\boldsymbol{s}_G^{(p)}(0, \cdot)$ is globally upper semicontinuous.

4. Increasing domains property. Let $(G_k)_{k=1}^\infty$ be sequence of domains in $\mathbb{C}^n$ such that $G_k \nearrow G$, i.e. $G_k \subset G_{k+1}$, $k \in \mathbb{N}$, and let $G = \bigcup_{k=1}^\infty G_k$. It is known that if $\mathcal{F} \in \{\mathcal{M}, \mathcal{K}\}$, then $d_{G_k}^{\mathcal{F}} \searrow d_G^{\mathcal{F}}$ and $\delta_{G_k}^{\mathcal{F}} \searrow \delta_G^{\mathcal{F}}$ (cf. [1, Propositions 2.7.1(a), 4.2.10(a)]). We will show that this is not true for $\mathcal{F} = \mathcal{S}^{(p)}$.

Example 4.1. Let

$$\varphi_k(\lambda) := \sum_{s=2}^{k} \frac{1}{s^2} \log\Big|\lambda - \frac{1}{s}\Big|, \quad k \ge 2, \quad \varphi(\lambda) := \sum_{s=2}^{\infty} \frac{1}{s^2} \log\Big|\lambda - \frac{1}{s}\Big|, \quad |\lambda| < \frac{1}{2}.$$

Observe that $\varphi_k \in \mathcal{PSH}$ and $\varphi_k \searrow \varphi$. Moreover, $\varphi_k \in \mathcal{C}^\infty(\frac{1}{k}\mathbb{D})$. Define

$$G_k := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1/2,\ |z_2| e^{\varphi_k(z_1)} < 1\},$$
$$G := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1/2,\ |z_2| e^{\varphi(z_1)} < 1\}.$$

Note that G_k is a Hartogs domain in $\mathbb{C}^2$, $k \ge 2$, and $G_k \nearrow G$. For each $k \ge 2$ the function $G_k \ni (z_1, z_2) \longmapsto (|z_2| e^{\varphi_k(z_1)})^{1+\varepsilon/p}$ belongs to $\mathcal{S}_{G_k}^{(p)}((0,0))$, $\varepsilon > 0$. Hence, $\boldsymbol{s}_{G_k}^{(p)}((0,0),(0,z_2)) \ge |z_2| e^{\varphi_k(0)} \ge |z_2| e^{\varphi(0)}$ for $|z_2| < e^{-\varphi(0)}$.

Analogously, since the function $G_k \ni (z_1, z_2) \longmapsto |z_2| e^{\varphi_k(z_1)}$ belongs to $\mathcal{S}_{G_k}^{(2p)}((0,0))$, we get $\boldsymbol{S}_{G_k}^{(2p)}((0,0);(0,X_2)) \ge |X_2| e^{\varphi(0)}$ for $X_2 \in \mathbb{C}$ and $k \ge 2$.

Now let $\sqrt[p]{u} \in \mathcal{S}_G^{(p)}((0,0))$. Since $\{1/s\} \times \mathbb{C} \subset G$, the Liouville type theorem for subharmonic functions gives $u(1/s, z_2) = \mathrm{const}(s) =: c_s$, $s \ge 2$, $z_2 \in \mathbb{C}$. Since $u(0,0) = 0$, we conclude that $c_s \longrightarrow 0$. Since u is continuous near $(0,0)$, we get $u(0, z_2) = \lim_{s\to+\infty} u(1/s, z_2) = \lim_{s\to+\infty} c_s = 0$, $|z_2| \ll 1$. Hence, since $\log u \in \mathcal{PSH}(G)$, we have $u(0, z_2) = 0$ for all $|z_2| < e^{\varphi(0)}$. Consequently, $\boldsymbol{s}_G^{(p)}((0,0),(0,z_2)) = 0$, $|z_2| < e^{\varphi(0)}$, and $\boldsymbol{S}_G^{(p)}((0,0);(0,X_2)) = 0$, $X_2 \in \mathbb{C}$.

5. Relations between $(\boldsymbol{m}_G, \boldsymbol{s}_G, \boldsymbol{g}_G)$ and $(\boldsymbol{\gamma}_G, \boldsymbol{S}_G, \boldsymbol{A}_G)$. We will discuss the following two problems. Find a pseudoconvex domain $G \subset \mathbb{C}^n$, $a \in G$, and $z^0 \in G$ (resp. $X_0 \in \mathbb{C}^n$) such that

$$\boldsymbol{m}_G(a, z^0) < \boldsymbol{s}_G(a, z^0) < \boldsymbol{g}_G(a, z^0)$$
$$(\text{resp. } \boldsymbol{\gamma}_G(a; X_0) < \boldsymbol{S}_G(a; X_0) < \boldsymbol{A}_G(a; X_0)).$$

Example 5.1. If $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$ are relatively prime, $\sigma(a) \ge 2$, and $\mu(a) \ge 2$, then the domain $G = \boldsymbol{D}_\alpha$ (cf. Example 3.3) is an example of a pseudoconvex domain (unfortunately, unbounded) such that $\boldsymbol{m}_G(a, \cdot) < \boldsymbol{s}_G(a, \cdot) < \boldsymbol{g}_G(a, \cdot)$

on $\boldsymbol{D}_\alpha \setminus \boldsymbol{V}_0$. It is not known whether there exists a bounded pseudoconvex domain with this property.

Example 5.2. Let $G \subset \mathbb{C}^n$ be a balanced domain (i.e. $\overline{\mathbb{D}} \cdot G = G$) and let $\mathfrak{h}_G(z)$ be the *Minkowski functional of* G. It is known that $G = \{z \in \mathbb{C}^n : \mathfrak{h}_G(z) < 1\}$. Moreover, $\boldsymbol{g}_G(0, \cdot) = \mathfrak{h}_G$ in $G \iff \boldsymbol{A}_G(0; \cdot) \equiv \mathfrak{h}_G \iff G$ is pseudoconvex $\iff \log \mathfrak{h}_G \in \mathcal{PSH}(\mathbb{C}^n)$ (cf. [1, Proposition 4.2.10(b)]).

Let $\widehat{G}$ be the convex envelope of G. It is known that $\widehat{G}$ is also balanced and $\mathfrak{h}_{\widehat{G}} = \sup\{q : q : \mathbb{C}^n \longrightarrow [0, +\infty)$ is a $\mathbb{C}$-seminorm with $q \le \mathfrak{h}_G\}$. Moreover (cf. [1, Proposition 2.3.1(d)]), $\boldsymbol{\gamma}_G(0; \cdot) \equiv \mathfrak{h}_{\widehat{G}}$. Thus, if G is pseudoconvex, then $\boldsymbol{\gamma}_G(0; \cdot) = \mathfrak{h}_{\widehat{G}} \ge \boldsymbol{S}_G(0; \cdot)$ and hence $\boldsymbol{\gamma}_G(0; \cdot) \equiv \boldsymbol{S}_G(0; \cdot) \equiv \mathfrak{h}_{\widehat{G}} \le \mathfrak{h}_G \equiv \boldsymbol{A}_G(0; \cdot)$. Consequently, we get the following result.

If G is a balanced pseudoconvex non-convex domain, then

$$\mathfrak{h}_{\widehat{G}} \equiv \boldsymbol{\gamma}_G(0; \cdot) \equiv \boldsymbol{S}_G(0; \cdot) \underset{\not\equiv}{\le} \boldsymbol{A}_G(0; \cdot) \equiv \mathfrak{h}_G.$$

In particular, the result solves the problem formulated in Example 4.2.17 from [1].

Example 5.3. Keep the notation from Example 3.1. Then

$$\boldsymbol{\gamma}_G(c_t; X_0) < \boldsymbol{S}_G^{(2p)}(c_t; X_0) = \boldsymbol{A}_G(c_t; X_0) = e^{\varphi(0,t)}, \quad p \in \mathbb{N},\ 0 < t \ll 1.$$

Indeed, the function $G \ni (z_1, z_2, z_3) \overset{v}{\longmapsto} |z_1| e^{\varphi(z_2, z_3)}$ is of the class $\mathcal{S}_G^{(2p)}(c_t)$, which gives

$$\boldsymbol{S}_G^{(2p)}(c_t; X_0) \ge \limsup_{\lambda \to 0} \frac{v(c_t + \lambda X_0)}{|\lambda|} = e^{\varphi(0,t)} > 0, \quad t > 0.$$

Observe that the mapping $e^{-\varphi(0,t)}\mathbb{D} \ni \lambda \overset{F}{\longmapsto} (\lambda, 0, t) \in G$ is well-defined. Hence, using the holomorphic contractibility, we get

$$\boldsymbol{A}_G(c_t; X_0) = \boldsymbol{A}_G(F(0); F'(0)(X_0)) \le \boldsymbol{A}_{e^{-\varphi(0,t)}\mathbb{D}}(0; 1) = e^{\varphi(0,t)}.$$

Thus, $\boldsymbol{S}_G^{(2p)}(c_t; X^0) = \boldsymbol{A}_G(c_t; X^0) = e^{\varphi(0,t)} \ge e^{\varphi(0,0)} > 0$, $t > 0$.

Now, to get the result it suffices to show that $\boldsymbol{\gamma}_G((0,0,0); X^0) = 0$ and then use the continuity of $\boldsymbol{\gamma}_G(\cdot; X^0)$. For this, let $f \in \mathcal{O}(G, \mathbb{D})$ such that $f(0,0,0) = 0$. Since $\{0\} \times \mathbb{C}^2 \subset G$, the Liouville theorem implies that $f(0, \cdot, \cdot) = \text{const}$. Since $f(0,0,0) = 0$, we get $f(0, \cdot, \cdot) \equiv 0$. Since $\mathbb{C} \times \{a_k\} \times \{0\} \subset G$, we get $f(\cdot, a_k, 0) = \text{const}(k)$. Thus, $f(\cdot, a_k, 0) \equiv 0$. Since the sequence $(a_k)_{k=1}^\infty$ is dense in $\mathbb{D}$, we conclude that $f = 0$ on $(\mathbb{C} \times \mathbb{D} \times \{0\}) \cap G$. Thus, $f(z_1, 0, 0) = 0$ provided that $|z_1| < e^{-\varphi(0,0)}$. Hence, $f'(0,0,0)(X^0) = 0$ and so $\boldsymbol{\gamma}_G((0,0,0); X^0) = 0$.

6. Derivative. Recall that for $\mathcal{F} \in \{\mathcal{M}, \mathcal{K}\}$ we have $\delta_G^{\mathcal{F}}(a; X) = \limsup_{\lambda \to 0} \frac{d_G^{\mathcal{F}}(a, a+\lambda X)}{|\lambda|}$, $a \in G$, $X \in \mathbb{C}^n$ (cf. (1.3) and (1.4)). It is an open problem whether

$$\boldsymbol{S}_G^{(2p)}(a; X) = \limsup_{\lambda \to 0} \frac{\boldsymbol{s}_G^{(2p)}(a, a + \lambda X)}{|\lambda|}, \quad a \in G,\ X \in \mathbb{C}^n.$$

Observe that

$$\boldsymbol{S}_G^{(2p)}(a;X) = \sup\Big\{\limsup_{\lambda\to 0}\frac{v(a+\lambda X)}{|\lambda|} : v \in \mathcal{S}_G^{(2p)}(a)\Big\}$$
$$\le \limsup_{\lambda\to 0}\frac{\boldsymbol{s}_G^{(2p)}(a,a+\lambda X)}{|\lambda|} \le \limsup_{\lambda\to 0}\frac{\boldsymbol{g}_G(a,a+\lambda X)}{|\lambda|} = \boldsymbol{A}_G(a;X),$$

so the problem is trivial if $\boldsymbol{S}_G^{(2p)}(a;X) = \boldsymbol{A}_G(a;X)$.

References

[1] Jarnicki, M., Pflug, P.: Invariant Distances and Metrics in Complex Analysis, 2nd extended edition, de Gruyter Expositions in Mathematics 9, Walter de Gruyter, xvii+861 pp (2013)

[2] Jarnicki, M., Pflug, P.: A remark on the Sibony function. J. Math. Anal. Appl. **461**, 1374–1377 (2018)

[3] Klimek, M.: Pluripotential Theory. Oxford University Press, Oxford (1991)

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